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Kelvin-wave diffraction by changes in depth

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The diffraction of a Kelvin wave by a transverse (to a straight coastline) discontinuity in depth is considered. A Fourier-integral formulation leads to a singular integral equation that may be solved exactly; however, the integrals in this solution are intractable without further approximation. An expansion to third order in a depth-change parameter yields results that are generally adequate for tidal problems (such as that posed by the Mendocino fracture zone) but are inadequate for the double-Kelvin-wave regime. Approximations are developed for a continuous change of depth that is either small or gradual, and the diffracted Kelvin wave along the coastline is found to have an amplitude that is inversely proportional to the square root of the depth and a phase that is given by the integral of the wavenumber.

1. Introduction

The following investigation of Kelvin-wave diffraction by a change in depth (in the direction of propagation) forms part of a broader study of coastal propagation of the tides and complements an earlier study (Miles 1972) in which the effects on a Kelvin wave of the Earth's curvature, the reduction in depth over the continental shelf, and distortion of the (otherwise straight) boundary were considered. It is aimed especially at the Mendocino fracture zone, where the change is sufficiently abrupt (at tidal wavelengths) to be modelled by a discontinuity in a semi-infinite ocean bounded by a vertical wall (see figure 1). We consider this model first and then go on to consider the more general case of continuously varying depth, subject to the restriction that the total change is either small or gradual (slowly varying). The problem of small, but otherwise arbitrary, changes in depth along a coastline has been considered previously by Pinsent (1972); however, both his analysis and his results contain errors (see § 5 below).

Let $\zeta(x,y)$ be a complex amplitude of a harmonic disturbance, such that the instantaneous vertical displacement of the free surface is given by

$$\hat{\zeta}(x,y,t) = \mathscr{R}\{\zeta(x,y)\,e^{i\sigma t}\}, \tag{1.1}$$

where $\mathscr{R}$ implies *the real part of* and σ is the angular frequency. The complex amplitude of the incident Kelvin wave is

$$\zeta = A_+\,e^{-ax+iky}, \tag{1.2}$$

where
$$\{a,k\} = (gh)^{-\frac{1}{2}}\{f,\sigma\}, \tag{1.3}$$

$\frac{1}{2}f$ is the vertical component of the angular velocity of the Earth, positive in the sense of figure 1, and h is the depth outside the continental shelf (we append the subscripts $\pm$ to h and h-dependent parameters if and as required to avoid ambiguity). The transmitted Kelvin wave is described by a similar expression, with A_+ replaced by A_-, and we seek the complex transmission coefficient

$$T = A_-/A_+ \tag{1.4}$$

as a function of the parameters

$$\epsilon = (h_+ - h_-)/(h_+ + h_-), \tag{1.5}$$

which measures the total change in depth, and

$$\not{f} = f/\sigma, \tag{1.6}$$

which measures the wave period in half pendulum-days.

For analytical convenience, we assume that

$$\arg \sigma = -\delta \quad (0 < \delta \ll 1). \tag{1.7}$$

The assumption that $\mathscr{R}\sigma > 0$ imposes no essential restriction on the results. The assumption that $\mathscr{I}\sigma < 0$ stems from a consideration of the corresponding initial-value problem and is tantamount to a radiation condition in the limit $\delta \downarrow 0$; this limit is implicit ($\delta = 0+$) in the ultimate interpretation of the results, and a statement such as $\not{f} > 1$ implies $\mathscr{R}\not{f} > 1$ in those stages of the development in which $\delta > 0$. The extension of the results to $f < 0$ is straightforward and requires only appropriate changes of sign, including those in the exponent of (1.2).

Additional (to the incident and diffracted Kelvin waves) disturbances are excited by the discontinuity and are necessary to satisfy the boundary conditions; however, these disturbances radiate energy away from the coastline only for certain ranges of ϵ and $\not{f}$. Poincaré waves of asymptotic form (as $r \to \infty$)

$$\zeta_P \sim P(\theta)\, r^{-\frac{1}{2}} \exp\{-i(k^2 - a^2)^{\frac{1}{2}} r\} \quad (r \to \infty) \tag{1.8}$$

are radiated if $\not{f} < 1$. A double Kelvin wave of the form (Longuet-Higgins 1968)

$$\zeta = D \exp\{-imx - (m^2 + a^2 - k^2)^{\frac{1}{2}} |y|\} \quad (x \to \infty), \tag{1.9}$$

where m is the positive real root of

$$h_+(m^2 + a_+^2 - k_+^2)^{\frac{1}{2}} + h_-(m^2 + a_-^2 - k_-^2)^{\frac{1}{2}} - \not{f}(h_+ - h_-)\, m = 0, \tag{1.10}$$

is propagated along the discontinuity if and only if $\epsilon\not{f} > 1$. The parametric regimes implied by these considerations are: (i) $0 < \not{f} < 1$, Kelvin and Poincaré waves; (ii) $-1 < \epsilon < 1/\not{f} < 1$, Kelvin waves; (iii) $1/\not{f} < \epsilon < 1$, Kelvin and double Kelvin waves. The regime of principal interest for tidal problems is roughly $|\epsilon| < \frac{1}{4}$ and $\not{f} < 2$, so that double Kelvin waves are only of peripheral interest in the present context. [The double Kelvin wave excited in an infinite ocean with a semi-infinite barrier along a plane discontinuity in depth has been calculated by Pinsent (1971).]

The fact that no energy can be radiated away from the coastline for

$$-1 < \epsilon < 1/\not{f} < 1$$

implies that the amplitude of the diffracted Kelvin wave must be inversely proportional to the square root of the depth (the energy flux for a Kelvin wave is proportional to $cA^2/a = ghA^2/\sigma$, where c is the wave speed and $1/a$ is the transverse scale of the trapped wave); accordingly

$$|T| = (h_+/h_-)^{\frac{1}{2}} = \{(1+\epsilon)/(1-\epsilon)\}^{\frac{1}{2}} \equiv T_0 \quad (-1 < \epsilon < 1/f < 1). \tag{1.11}$$

Moreover, the energy radiated by Poincaré waves is of $O(\epsilon^2)$, so that (1.11) also holds to $O(\epsilon)$ for $|\epsilon| \ll 1$ and $f < 1$. In fact, the results in §4 below imply that (1.11) provides an excellent approximation throughout the tidal regime; e.g. $|T/T_0| = 1{\cdot}002$ for the semi-diurnal tide at the Mendocino fracture zone.

Energy arguments are inadequate for an estimate of the phase of T; however, heuristic arguments suggested by Green's approximation for one-dimensional gravity waves in shallow water of slowly varying depth (Lamb 1932, §185) suggest that this change must be of $O(\epsilon^2)$. This is borne out by the results in §4 below, which yield $\arg T < 0{\cdot}8\epsilon^2$ for $f < 2$. Taken together, these arguments suggest that a first-order (in ϵ) approximation should be adequate, and that

$$\zeta(0,y) = A_0\{h(y)/h_0\}^{-\frac{1}{2}} \exp\left\{i\int_0^y k\,dy\right\} \tag{1.12}$$

should provide a good approximation, for a Kelvin wave moving through water of continuously varying depth. This conjecture is supported by analysis in §5 below, where it is shown to be valid within $1+O(\epsilon^2)$ as $|\epsilon| \to 0$ or, alternatively, as an asymptotic approximation, similar to that of Green, for slowly varying depth (scattering is negligible in the latter approximation).

2. The boundary-value problem

Let ζ, u and v denote the complex amplitudes of the vertical displacement of the free surface and of the x and y components of the particle velocity in the rotating reference frame of figure 1. The equations of motion for small disturbances in shallow water of uniform depth h then imply (Lamb 1932, §207)

$$\{u, v\} = (ig/\sigma)(1-f^2)^{-1}\{\zeta_x - if\zeta_y, \zeta_y + if\zeta_x\} \tag{2.1}$$

and

$$\zeta_{xx} + \zeta_{yy} + \kappa^2\zeta = 0 \quad (y \neq 0), \tag{2.2}$$

where

$$\kappa^2 = (\sigma^2 - f^2)/(gh) \equiv k^2 - a^2. \tag{2.3}$$

They also imply

$$f\zeta + h(u_y - v_x) = 0, \tag{2.4}$$

which describes (for small disturbances) the conservation of vorticity in a column of depth $h+\zeta$ and infinitesimal cross-section. [The term $-if(h_x u + h_y v)$ must be added to the left-hand side of (2.4) if $h = h(x,y)$.]

The differential equation (2.2) is not valid at $y = 0$, where it must be replaced by the requirements that pressure and transverse mass flux be continuous across the discontinuity in h;

$$[\zeta] = 0, \quad [hv] = 0, \tag{2.5a, b}$$

where

$$[\;] = [\;]_{y=0+} - [\;]_{y=0-}.$$

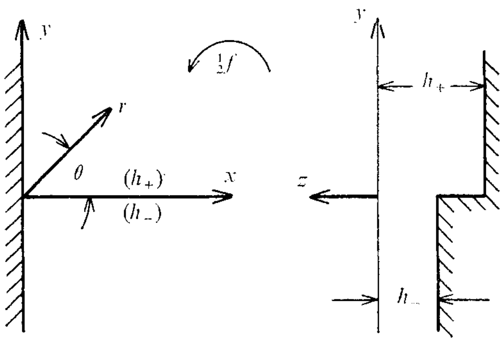

FIGURE 1. Discontinuity in depth along $y = 0$ in a semi-infinite ocean bounded by $x = 0$.

Combining (2.4) and (2.5 a, b), we obtain the alternative jump condition

$$[hu_y] = 0 \quad (y = 0), \tag{2.5c}$$

which proves more convenient than (2.5 b) in the subsequent formulation.

We require the solution of (2.2) and (2.5) subject to the boundary conditions

$$u = 0 \quad (x = 0), \qquad \zeta \to 0 \quad (x \to \infty), \tag{2.6a, b}$$

and

$$\zeta \sim A_+ e^{-ax+iky} \quad (y \to \infty, ax = O(1)), \tag{2.6c}$$

where the last term represents the incident Kelvin wave of (1.2). The invocation of radiation conditions is unnecessary by virtue of (1.7).

3. Fourier-integral formulation

The most general solution of (2.2) that satisfies (2.6 a, b, c) may be posed in the form

$$\zeta(x,y) = A_\pm e^{-ax+iky} + \frac{2i}{\pi}\int_0^\infty U_\pm(\alpha)\, e^{-\mu|y|}\left(\frac{\alpha\cos\alpha x \mp i f\mu \sin\alpha x}{\alpha^2+a^2}\right) d\alpha \quad (y \gtrless 0), \tag{3.1}$$

where

$$\mu = (\alpha^2 - \kappa^2)^{\frac{1}{2}} \quad (\mathscr{R}\mu \geqslant 0), \tag{3.2}$$

the subscripts $\pm$ are implicit on the h-dependent parameters a, k, κ and μ for $y \gtrless 0$, and the path of integration (that part of C_0 in $\mathscr{R}\alpha > 0$ in figure 2) is indented over the branch points at $\alpha = \kappa_\pm$ in the limit $\delta \downarrow 0$. Substituting (3.1) into (2.1) yields

$$(\sigma/g)u \equiv v(x,y) = (2/\pi)\int_0^\infty U_\pm(\alpha)\, e^{-\mu|y|} \sin\alpha x \, d\alpha, \tag{3.3}$$

whence

$$U_\pm(\alpha) = \int_0^\infty v(x, 0\pm) \sin\alpha x \, dx \tag{3.4}$$

is the Fourier-sine transform of the dimensionless velocity v on $y = 0\pm$.

It remains to satisfy the jump conditions at $y = 0$. Substituting (3.3) into (2.5 c), we obtain

$$h_+\mu_+U_+ = -h_-\mu_-U_- \equiv -iA_+\alpha\Phi(\alpha), \tag{3.5}$$

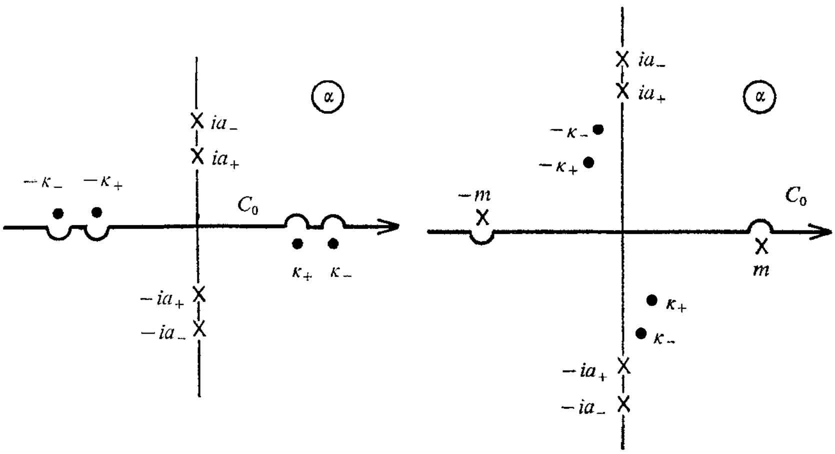

FIGURE 2. The complex-α plane, showing the disposition of the branch points (●) and poles (×) of $\Phi(\alpha)$ for $0 < \delta \ll 1$ and $\epsilon > 0$. The branch points tend to the real/imaginary axis as $\delta \downarrow 0$ with $f \lessgtr 1$. The poles at $\alpha = \pm m$ appear in the cut plane (in which the cuts are determined by the requirement $\mathscr{R}\mu \geqslant 0$) if and only if $|\epsilon f| > 1$. The positions of the singular points with + and − subscripts coincide in the limit $\epsilon \to 0$ and are interchanged if $\epsilon < 0$.

where (since $U_\pm$ is an odd function of α) Φ is an even function of α. Substituting $U_\pm$ from (3.5) into (3.1) and invoking (2.5*a*) and (1.4), we obtain

$$(2/\pi)\int_0^\infty \Phi(\alpha)\{F_e(\alpha)\cos\alpha x + iF_o(\alpha)\sin\alpha x\}\,d\alpha = T\,e^{-a_- x} - e^{-a_+ x}, \tag{3.6}$$

where $$F_e(\alpha) = \alpha^2[\{h_+(\alpha^2+a_+^2)\,\mu_+\}^{-1} + \{h_-(\alpha^2+a_-^2)\,\mu_-\}^{-1}] \tag{3.7a}$$

and $$F_o(\alpha) = f\alpha[-\{h_+(\alpha^2+a_+^2)\}^{-1} + \{h_-(\alpha^2+a_-^2)\}^{-1}] \tag{3.7b}$$

are the even and odd parts of

$$F(\alpha) \equiv F_e(\alpha) + F_o(\alpha) \tag{3.8a}$$

$$= \frac{(1-f^2)\,\alpha^2\{h_+\mu_+ + h_-\mu_- + f(h_+ - h_-)\,\alpha\}}{h_+h_-\mu_+\mu_-(\alpha+f\mu_+)\,(\alpha - f\mu_-)}. \tag{3.8b}$$

Taking the Fourier-cosine transform of (3.5), we obtain [the transform of the integral involving $\sin\alpha x$ may be calculated by replacing α by s in the integrand, multiplying through by $\exp(-cx)\cos\alpha x$, integrating from $\alpha = 0$ to $\alpha = \infty$, letting $c \downarrow 0$, and invoking symmetry considerations to reduce the integral to that displayed in (3.9)]

$$F_e(\alpha)\,\Phi(\alpha) + \frac{i}{\pi}\,⨍_{-\infty}^{\infty} \frac{F_o(s)\,\Phi(s)\,ds}{s-\alpha} = \frac{Ta_-}{\alpha^2+a_-^2} - \frac{a_+}{\alpha^2+a_+^2}, \tag{3.9}$$

where the crossed integral sign implies the Cauchy principal value of the integral.† Letting $\alpha \to 0$ in (3.9) and invoking (3.7*a*) and the condition

$$\alpha^2\Phi(\alpha) \to 0 \quad (\alpha \to 0), \tag{3.10}$$

† The Fourier-sine transform of (3.6) is equivalent to the Hilbert transform of (3.9).

which is necessary for the evanescence of the Fourier integral (3.4) as $x \to \infty$, we obtain

$$T = T_0 + (ia_-/\pi) \int_{-\infty}^{\infty} s^{-1} F_o(s)\, \Phi(s)\, ds, \tag{3.11}$$

where

$$T_0 = a_-/a_+ = (h_+/h_-)^{\frac{1}{2}} \tag{3.12}$$

is the transmission coefficient defined by (1.11). Substituting (3.11) into (3.9), combining the two integrals, invoking symmetry considerations to simplify the resulting integral, and multiplying the result through by $(a_-^2 + \alpha^2)/\alpha^2$, we obtain

$$G_e(\alpha)\, \Phi(\alpha) + (i/\pi) \int_{-\infty}^{\infty} (s-\alpha)^{-1} G_o(s)\, \Phi(s)\, ds = E(\alpha), \tag{3.13}$$

where $G_e(\alpha)$ and $G_o(\alpha)$ are the even and odd parts of

$$G(\alpha) = \alpha^{-2}(\alpha^2 + a_-^2) F(\alpha) \tag{3.14a}$$

$$= \left(\frac{\alpha + \ell\mu_-}{\alpha + \ell\mu_+}\right) \left\{\frac{h_+\mu_+ + h_-\mu_- + \ell(h_+ - h_-)\,\alpha}{h_+ h_- \mu_+ \mu_-}\right\}, \tag{3.14b}$$

and

$$E(\alpha) = \{(a_-^2/a_+) - a_+\}(\alpha^2 + a_+^2)^{-1} \tag{3.15a}$$

$$= 2\epsilon(1-\epsilon)^{-1} a_+ (\alpha^2 + a_+^2)^{-1}. \tag{3.15b}$$

We remark that $G(\alpha)$ has a simple zero at $\alpha = \mp m \pm i0+$ $(\delta = 0+)$ on the real axis if and only if $\pm\epsilon\ell > 1$; m is given by (1.10).

The solution of (3.13) may be reduced to quadrature (see appendix), but it does not appear possible to obtain tractable results without further approximation.

4. Approximate solution

If $\epsilon \ll 1$ and $\ell = O(1)$, as is true for most tidal problems but *not* for the double-Kelvin-wave regime ($|\epsilon\ell| > 1$), the solution of (3.13) may be obtained by expanding G and Φ in powers of ϵ. Introducing

$$h_\pm = h_1(1 \pm \epsilon), \quad h_1 = \tfrac{1}{2}(h_+ + h_-), \quad \epsilon = (h_+ - h_-)/(h_+ + h_-), \tag{4.1 a, b, c}$$

in (3.8) and (3.13)–(3.15), we obtain

$$F(\alpha) = 2\alpha^2(\alpha^2 + a_1^2 + \epsilon\ell\alpha\mu_1)(h_1\mu_1)^{-1}(\alpha^2 + a_1^2)^{-2}\{1 + O(\epsilon^2)\}, \tag{4.2}$$

and

$$\Phi(\alpha) = \frac{\epsilon h_+ a_+ \mu_1}{\alpha^2 + a_1^2}\left\{1 - \frac{2i\epsilon\ell(\alpha^2 + a_1^2)}{\pi} \int_0^{\infty} \frac{s^2\mu_1(s)\, ds}{(s^2 + a_1^2)^2 (s^2 - \alpha^2)} + O(\epsilon^2)\right\}, \tag{4.3}$$

where a_1 and μ_1 are based on h_1.

Substituting (4.2) and (4.3) into (3.11), we obtain

$$\frac{T}{T_0} = 1 + \frac{4i\epsilon^2\ell a_1^2}{\pi}\int_0^{\infty} \frac{\alpha^2\mu_1\, d\alpha}{(\alpha^2 + a_1^2)^3} + \frac{8\epsilon^3\ell^2 a_1^2}{\pi}\int_0^{\infty} \frac{\alpha^2\mu_1(\alpha)\, d\alpha}{(\alpha^2 + a_1^2)^2} \int_0^{\infty} \frac{s^2\mu(\alpha)\, ds}{(s^2 - \alpha^2)(s^2 + a_1^2)^2} + O(\epsilon^4), \tag{4.4}$$

in which the double integral vanishes identically. Evaluating the first integral, we obtain

$$|T/T_0| = 1 - \tfrac{1}{4}\epsilon^2(1 - \ell^2)^2 H(1 - \ell) + O(\epsilon^4) \tag{4.5}$$

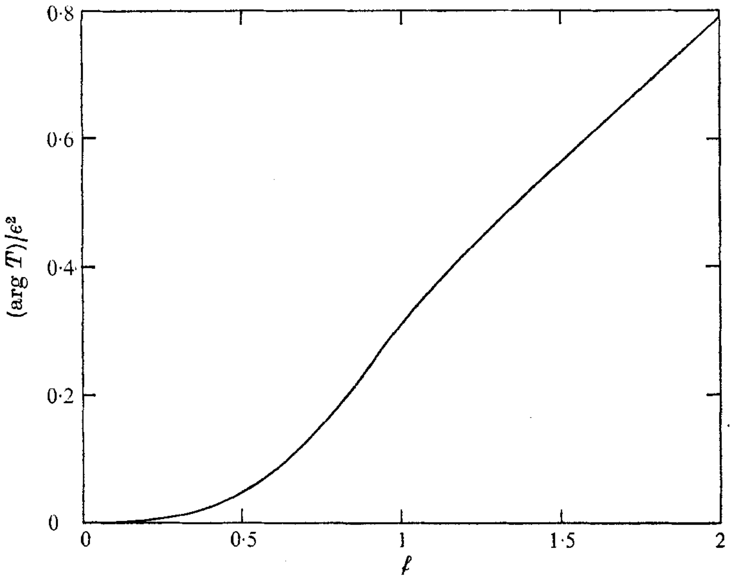

FIGURE 3. The phase of the Kelvin-wave transmission coefficient for a discontinuity in depth, as given by (4.6*a*).

and
$$\arg T = \tfrac{1}{2}(\epsilon^2/\pi)\{\ell(1+\ell^2)-\tfrac{1}{2}(1-\ell^2)^2\ln|(1+\ell)/(1-\ell)|\}+O(\epsilon^4) \qquad (4.6a)$$
$$= 4(3\pi)^{-1}\epsilon^2\ell^3\{1-\tfrac{1}{5}\ell^2+O(\ell^4)\} \quad (\ell\downarrow 0). \qquad (4.6b)$$

The result $|T| = T_0$ is exact for $1 < \ell < 1/\epsilon$; however, our approximations are valid only for $\epsilon\ell \ll 1$. The result (4.6*a*) is plotted in figure 3. Considering, for example, the Mendocino fracture zone (40° N), for which $\epsilon = \frac{1}{7}$ ($h_+ \doteqdot 4$ km, $h_- \doteqdot 3$ km), we obtain $T_0 = 1{\cdot}15$, $|T/T_0| = 1$ and $1{\cdot}0017$, and $\arg T = 0{\cdot}096$ and $0{\cdot}0022$ for the diurnal (K_1) and semidiurnal (M_2) tides. The corresponding time advances are 22 min and 15 s, which compare with a time delay of 1·3 h for either component in consequence of the corner at Cape Mendocino (Miles 1972).

Turning to the calculation of the non-Kelvin-wave field, we rest content with a first-order (in ϵ) approximation. Higher approximations are qualitatively similar if $|\epsilon\ell| < 1$, but are complicated by the distinction between the two sets ($h = h_\pm$) of singularities. Substituting the first approximation to Φ from (4.3) into (3.3), dropping the subscripts (since the parameters for $y > 0$ and $y < 0$ are equal in this first approximation) and setting $A_+ = 1$, we obtain

$$v = -(2i\epsilon a/\pi)\,\mathrm{sgn}\,y\int_0^\infty \alpha(\alpha^2+a^2)^{-1}\,e^{-\mu|y|}\sin\alpha x\,d\alpha \qquad (4.7a)$$
$$= -(\epsilon a/\pi)\,\mathrm{sgn}\,y\int_{-\infty}^\infty \alpha(\alpha^2+a^2)^{-1}\,e^{i\alpha x-\mu|y|}\,d\alpha, \qquad (4.7b)$$

where, here and subsequently, the error is $O(\epsilon^2)$. Letting $|y| \to 0$, we obtain
$$v = -i\epsilon a\,e^{-ax}\,\mathrm{sgn}\,y \quad (x>0,\ |y|\to 0), \qquad (4.8)$$
which describes the discontinuity in tangential velocity at the discontinuity in depth. The integrals in (4.7*a*, *b*) do not converge uniformly near $x = y = 0$, in consequence of which (4.8) satisfies (2.6*a*) only if averaged over $y = 0\pm$.

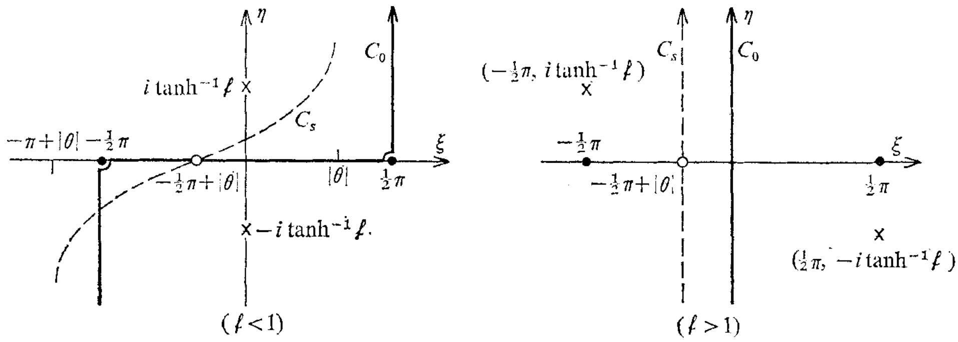

FIGURE 4. The complex-ζ plane, showing the disposition of the various branch points ($\bullet$), poles ($\times$), and saddle points ($\bigcirc$), the original paths of integration (C_0), and the paths of steepest descent (C_s) for $\delta = 0+$.

We obtain an asymptotic approximation to v by the method of steepest descent (cf. Voit 1958). Considering first $\ell < 1$, we introduce the transformations

$$\alpha = \kappa \sin\zeta, \quad \mu = i\kappa\cos\zeta \quad (\zeta = \xi + i\eta) \tag{4.9}$$

and

$$x = r\cos\theta, \quad y = r\sin\theta \tag{4.10}$$

in (7.4*b*) to obtain

$$v = -(\epsilon a/\pi)(1-\ell^2)\operatorname{sgn} y \int_{-\frac{1}{2}\pi - i\infty}^{\frac{1}{2}\pi+i\infty} \frac{\sin\zeta\cos\zeta\exp\{i\kappa r\sin(\zeta-|\theta|)\}\,d\zeta}{\sin^2\zeta+\ell^2\cos^2\zeta}, \tag{4.11}$$

where the path of integration C_0 is sketched in figure 4. The exponent in the integrand has a saddle point at $\zeta = -\frac{1}{2}\pi + |\theta|$, and the path of steepest descent through that saddle point for $\ell < 1$, say C_s, is given by

$$\sin(\xi - |\theta|)\cosh\eta = -1 \quad (\ell < 1, \delta = 0+), \tag{4.12}$$

as sketched in figure 4. Deforming C_0 to C_s, we find that the pole at $\alpha = ia$ is crossed if and only if $|\theta| < \cos^{-1}\ell$ (the pole at $\alpha = -ia$ is not crossed for any value of θ). Carrying out a saddle-point approximation to the integral over C_s, we obtain

$$v \sim \epsilon a[-i(\operatorname{sgn} y)H(\cos^{-1}\ell - |\theta|)\,e^{-ax-ik|y|} + (1-\ell^2)(\tfrac{1}{2}\pi\kappa r)^{-\frac{1}{2}}\,e^{-i(\kappa r - \frac{1}{4}\pi)}$$
$$\times(\cos^2\theta + \ell^2\sin^2\theta)^{-1}\sin\theta\cos\theta] \quad (\kappa r\to\infty, \ell < 1), \tag{4.13}$$

where H is Heaviside's step function. The asymptotic approximation is coincidentally exact at $|y| = |\theta| = 0$, where (4.13) reduces to (4.8); on the other hand, it is not uniformly valid near $\theta = \cos^{-1}\ell$.

If $\ell > 1$, κ must be replaced by $-i|\kappa|$ (since $\arg \ell = 0+$ for $\delta = 0+$) in (4.9) and (4.11), C_0 transforms to $\xi = 0$, and C_s is given by $\xi = -\frac{1}{2}\pi + |\theta|$. The pole at $\alpha = ia$ then is not crossed in the deformation from C_0 to C_s if $|\theta| > 0$, and (4.13) is replaced by

$$v \sim -i\epsilon a(\ell^2-1)(\tfrac{1}{2}\pi|\kappa|\,r)^{-\frac{1}{2}}\,e^{-|\kappa|r}$$
$$\times(\cos^2\theta + \ell^2\sin^2\theta)^{-1}\sin\theta\cos\theta \quad (|\kappa|\,r\to\infty, |\theta| > 0, \ell > 1). \tag{4.14}$$

The approximation is not uniformly valid near $|\theta| = 0$.

The asymptotic calculation of the non-Kelvin-wave displacement field, say ζ_*, differs qualitatively from the preceding calculation only in that the integrand in (3.1) has a double pole at $\alpha = ia$ if $y < 0$. The end results are

$$\zeta_*(x,y) \sim \epsilon H(\cos^{-1}f - |\theta|)\begin{Bmatrix} \frac{1}{2}(1-f^2) \\ -\frac{1}{2}(1+f^2)+a(x-ify) \end{Bmatrix} e^{-ax-ik|y|}$$

$$+\frac{\epsilon f(1-f^2)^{\frac{3}{2}}\, e^{-i(\kappa r-\frac{1}{4}\pi)}\sin\theta\cos\theta}{(\frac{1}{2}\pi\kappa r)^{\frac{1}{2}}(\cos\theta - if\sin\theta)(\cos^2\theta+f^2\sin^2\theta)} \quad (f<1,\ y \gtrless 0) \qquad (4.15a)$$

$$\sim \frac{-\epsilon f(f^2-1)^{\frac{3}{2}}\, e^{-|\kappa| r}\sin\theta\cos\theta}{(\frac{1}{2}\pi|\kappa|\,r)^{\frac{1}{2}}(\cos\theta - if\sin\theta)(\cos^2\theta+f^2\sin^2\theta)} \quad (f>1) \qquad (4.15b)$$

as $\kappa|r| \to \infty$ with $x > 0$, and

$$\zeta_*(0,y) \sim \epsilon f^{-3}(1-f^2)^{\frac{3}{4}}\,(\tfrac{1}{2}\pi|ky|^3)^{-\frac{1}{2}}\exp\{-i\kappa|y|+\tfrac{3}{4}i\pi\}\,\mathrm{sgn}\,y \quad (f<1) \qquad (4.16a)$$

$$\sim \epsilon f^{-3}(f^2-1)^{\frac{3}{4}}\,(\tfrac{1}{2}\pi|ky|^3)^{-\frac{1}{2}}\,e^{-|\kappa y|}\,\mathrm{sgn}\,y \quad (f>1). \qquad (4.16b)$$

5. Continuous variation of depth

We now suppose that

$$h = h_0\{1+\epsilon \hbar(y)\}, \qquad (5.1a)$$

where

$$\hbar(0) = 0, \quad \hbar(y) = o(e^{-\delta|k|y}) \quad (y \to \infty). \qquad (5.1b, c)$$

The restriction (5.1c) is typically satisfied by $\hbar \equiv 0$ in $y \geqslant 0$ after an appropriate choice of origin; however, it is useful to have results for the weaker restriction that (for $\delta = 0+$) $\hbar$ must vanish smoothly as $y \to \infty$. The parameter ϵ is similar to that defined by (1.5), but may be regarded as positive.

The equations of motion for continuously varying h yield

$$(1+\epsilon\hbar)(\zeta_{xx}+\zeta_{yy})+\epsilon\hbar'(\zeta_y+if\zeta_x)+\kappa_0^2\zeta = 0 \qquad (5.2)$$

in place of (2.2), where the prime implies differentiation with respect to y, and the subscript zero implies $h = h_0$. We seek that solution of (5.2) and the boundary conditions (2.6a, b) which reduces to

$$\zeta_0(x,y) = e^{-a_0x+ik_0y} \qquad (5.3)$$

for $\epsilon = 0$.

Considering first a perturbation approximation, we substitute

$$\zeta(x,y) = \zeta_0(x,y)+\epsilon\zeta_1(x,y)+O(\epsilon^2) \qquad (5.4)$$

into (5.2) and (2.6a, b) to obtain

$$(\partial_x^2+\partial_y^2+\kappa_0^2)\,\zeta_1 = -\hbar(\partial_x^2+\partial_y^2)\,\zeta_0 - \hbar'(\partial_y+if\partial_x)\,\zeta_0 \qquad (5.5a)$$

$$= -i(1-f^2)\,k_0(\hbar'+ik_0\hbar)\,\zeta_0 \qquad (5.5b)$$

and

$$(\partial_x - if\partial_y)\,\zeta_1 = 0 \quad (x=0), \qquad \zeta_1 \to \infty \quad (x \to \infty). \qquad (5.6a, b)$$

Extending the expansion to $\epsilon^n\zeta_n$ would yield (5.5a) and (5.6a, b), but not (5.5b), with ζ_1 and ζ_0 replaced by ζ_n and ζ_{n-1}.

Solving (5.5b) and (5.6a, b) through Fourier transformation with respect to y and dropping the subscript zero on a and k (which we may do without ambiguity in the calculation of ζ_1), we obtain

$$\zeta_1(x,y) = \int_{-\infty}^{\infty} \mathscr{G}(x, y-\eta)\, e^{ik\eta}\{\hbar'(\eta) + ik\hbar(\eta)\}\, d\eta, \tag{5.7}$$

where
$$\mathscr{G}(x,y) = \frac{i(1-\ell^2)\,k}{2\pi}\int_{-\infty}^{\infty}\left\{e^{-ax} - \ell\left(\frac{k-\beta}{\nu-\ell\beta}\right)e^{-\nu x}\right\}\frac{e^{i\beta y}\,d\beta}{\beta^2-k^2}, \tag{5.8}$$

$$\nu = (\beta^2-\kappa^2)^{\frac{1}{2}} \quad (\mathscr{R}\nu \geqslant 0), \tag{5.9}$$

and the path of integration is along the real axis of the complex-β plane if $\delta > 0$. The complete integrand has a simple pole at $\beta = \kappa$ but is regular at $\beta = -k$.

The integrals in (5.7) and (5.8) may be transformed in various ways. We consider first the important special case $x = 0$. Deforming the path of integration for $\mathscr{G}$ into $\mathscr{I}\beta \gtrless 0$ for $y-\eta \gtrless 0$, and separating out the contribution of the pole at $\beta = k$ if $y-\eta < 0$, we obtain

$$\mathscr{G}(0,y) = \tfrac{1}{2}e^{iky}H(y) + \frac{i\ell(1-\ell^2)\,k}{2\pi}\int_{C_\pm}\frac{e^{i\beta y}\,d\beta}{(\nu-\ell\beta)(\beta+k)}, \tag{5.10}$$

where $C_\pm$ passes counterclockwise/clockwise around the vertical branch cut from $\beta = \mp\kappa$ to $\beta = \mp\kappa \pm i\infty$. Substituting (5.10) into (5.7), invoking the identity

$$(1-\ell^2)/(\nu-\ell\beta) = (\nu+\ell\beta)/(\beta^2-k^2),$$

integrating the term in $ik\hbar(\eta)$ by parts, invoking (5.1c), and introducing the change of variable $\beta = \mp\kappa \pm it$ along $C_\pm$, we obtain

$$\zeta_1(0,y) = \tfrac{1}{2}e^{iky}\left\{-\hbar(y) + ik\int_y^{\infty}\hbar(\eta)\,d\eta\right\} + \zeta_{1*}(y), \tag{5.11}$$

where
$$\zeta_{1*}(y) = \frac{i\ell k}{2\pi}\int_{-\infty}^{\infty}\hbar'(\eta)\,d\eta\int_{C_\pm}\frac{\beta\nu\, e^{i\beta(y-\eta)}\,d\beta}{(\beta^2-k^2)^2} \tag{5.12a}$$

$$= -(\ell k/\pi)\int_{-\infty}^{\infty}\hbar'(\eta)\exp\{-i\kappa|y-\eta| - \tfrac{1}{4}i\pi\}\,\mathrm{sgn}\,(y-\eta)\,d\eta$$
$$\times\int_0^{\infty} t^{\frac{1}{2}}(2\kappa-it)^{\frac{1}{2}}(\kappa-it)\{k^2-(\kappa-it)^2\}^{-2}e^{-t|y-\eta|}\,dt. \tag{5.12b}$$

Substituting (5.11) into (5.4), restoring the subscript zero on k, and observing that

$$1 - \tfrac{1}{2}\epsilon\hbar(y) = \{h(y)/h_0\}^{-\frac{1}{4}} = k(y)/k_0 \tag{5.13}$$

to first order in ϵ, we obtain

$$\zeta(0,y) = \{h(y)/h_0\}^{-\frac{1}{4}}\exp\left\{i\int_0^y k\,dy - i\int_0^{\infty}(k-k_0)\,dy\right\} + \epsilon\zeta_{1*}(y) + O(\epsilon^2). \tag{5.14}$$

The dominant term in (5.14) represents the Kelvin wave and is equivalent to the approximation obtained by invoking $T = T_0$ in the calculation of the diffracted Kelvin wave in §§3 and 4. The corresponding approximation derived by Pinsent (1972, equation (3.11)) implies the diffracted Kelvin wave

$$\zeta_K(0,y) = e^{ik_0 y}\left\{1 + \tfrac{1}{2}ik_0\,\epsilon\int_{-\infty}^{\infty}\hbar\,dy + O(\epsilon^2)\right\} \tag{5.15}$$

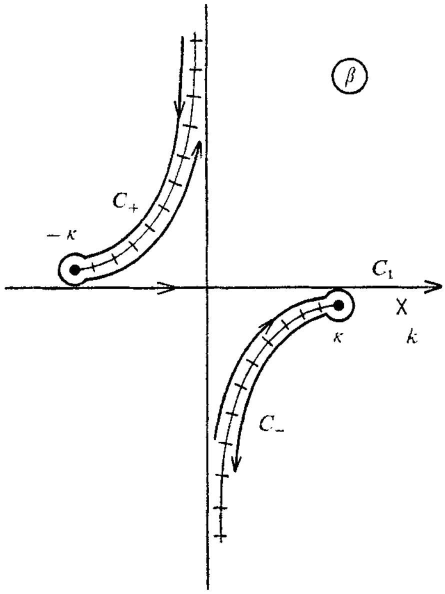

FIGURE 5. The contours along $\mathscr{R}\nu = 0+$, which map onto $\mathscr{I}\alpha = 0$ through the transformation (5.16).

in the present notation. This result is valid if and only if the change of depth is confined to a region that has been completely traversed (by the Kelvin wave) at the point of observation, in which case the Kelvin-wave component of (5.14) reduces to (5.15).

We transform the Fourier integral of (5.7) and (5.8) to one that is more closely related to the integrals in §§3 and 4 by: (i) integrating the term in $ik\hbar(\eta)$ by parts, which has the effect of replacing $\hbar' + ik\hbar$ by $\hbar'$ in (3.7) and introducing an additional factor of $\beta/(\beta-k)$ in (5.8); (ii) deforming the path of integration into $\mathscr{I}\beta \gtrless 0$ for $y-\eta \gtrless 0$; (iii) separating out the contribution of the double pole at $\beta = k$; (iv) introducing the change of variable

$$\beta = \pm i(\alpha^2-\kappa^2)^{\frac{1}{2}} \equiv \pm i\mu \quad (y \gtrless 0) \tag{5.16}$$

along a path around the cut from $\beta = \mp\kappa$ to $\beta = \pm\infty$ in the β plane ($C_\pm$ in figure 5), which maps on the real axis of the α plane (C_0 in figure 2, with $\kappa_+ = \kappa_- = \kappa$ therein); the end result is

$$\zeta_1(x,y) = \frac{1}{2}\left\{-\hbar(y) + ax\hbar(y) + ik\int_y^\infty \hbar(\eta)\,d\eta\right\} e^{-ax+iky} + \zeta_{1*}(x,y), \tag{5.17}$$

where

$$\zeta_{1*}(x,y) = \frac{(1-f^2)a}{2\pi}\int_{-\infty}^{\infty} \hbar'(\eta)\,\mathrm{sgn}\,(y-\eta)\,d\eta \int_{-\infty}^{\infty} \frac{e^{i\alpha x-\mu|y-\eta|}\,\alpha\,d\alpha}{(\alpha^2+a^2)\{\alpha+f\mu\,\mathrm{sgn}\,(y-\eta)\}}. \tag{5.18}$$

Substituting (5.17) into (5.4), restoring the subscript zero on a and k, and invoking (5.13) and the corresponding result for $a(y)$, we obtain

$$\zeta(x,y) = \{h(y)/h_0\}^{-\frac{1}{4}}\exp\left\{-a(y)\,x + i\int_0^y k\,dy - i\int_0^\infty (k-k_0)\,dy\right\} + \epsilon\zeta_{1*}(x,y) + O(\epsilon^2). \tag{5.19}$$

The dominant term in (5.19) may again be identified as the diffracted Kelvin wave; however, this identification is unambiguous only on $x = 0$, since ζ_{1*} contains a Kelvin-wave-like component in $x > 0$, as exemplified by the first term on the right-hand side of (4.15*a*). An asymptotic approximation to the scattered wave defined by (5.18) may be obtained by invoking the transformations (4.9) and (4.10) and the method of steepest descent. The results reduce to (4.15) and (4.16) for the depth profile (4.1), which implies $h_0 = h_+$ and

$$\hbar(y) = -2H(-y)$$

in the notation of (5.1).

An alternative approach to the solution of (5.2) and (2.6*a*, *b*) is to assume that $h(y)$ is slowly varying ($|h'| \ll kh$). Posing the solution in the form

$$\zeta(x,y) = A(x,y)\exp\left\{-a(y)\,x + i\int_0^y k\,dy\right\}, \qquad (5.20)$$

expanding A in powers of x, and letting $k \to \infty$ with $kx = O(1)$, we obtain the asymptotic solution (cf. Miles 1972, § 3)

$$A = \text{constant} \times \{k(y) + i f x k'(y) - \tfrac{1}{2} i x^2 k(y)\,k'(y) + O(1/k)\}. \qquad (5.21)$$

Substituting (5.21) into (5.20) and setting $x = 0$, we place the result in the form

$$\zeta(0,y) \sim A_0(h/h_0)^{-\frac{1}{2}}\exp\left(i\int_0^y k\,dy\right), \qquad (5.22)$$

which is equivalent to the Kelvin wave in (5.14), but is subject to the restriction $|h'| \ll kh$ rather than $\epsilon \ll 1$.

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Appendix. Solution of singular integral equation

The singular integral equation (3.13) is of a type considered by Muskhelishvili (1953, § 108), in whose notation $G_e = A$, $G_o = B$, $E = g$ and $\Phi = \psi$. The character of the solution depends essentially on the parameter ($a = \alpha$ in Muskhelishvili's notation)

$$a = \pm(2\pi)^{-1}\arg\{G(\mp\infty)/G(\pm\infty)\}, \qquad (\text{A }1a)$$

$$= (2\pi)^{-1}\arg\{(1-\epsilon f)/(1+\epsilon f)\}, \qquad (\text{A }1b)$$

$$= \begin{cases} \tfrac{1}{2} & (\epsilon f < -1) \\ 0 & (|\epsilon f| < 1) \quad (\delta = 0+). \\ -\tfrac{1}{2} & (\epsilon f > 1) \end{cases} \qquad (\text{A }1c)$$

The solution for $|\epsilon f| < 1$ is straightforward (the end-points $\pm\infty$ are 'special') and is given by

$$\Phi(\alpha) = K_e(\alpha) + \frac{1}{i\pi Z(\alpha)}\int_{-\infty}^{\infty}\frac{Z(s)\,K_0(s)\,ds}{s-\alpha}, \qquad (\text{A }2)$$

where $K_e(\alpha)$ and $K_o(\alpha)$ are the even and odd parts of

$$K(\alpha) = E(\alpha)/G(-\alpha), \tag{A 3a}$$

$$= \frac{2\epsilon(1-f^2)\,a_+h_+h_-\mu_+\mu_-}{(1-\epsilon)\,(\alpha+f\mu_+)\,(\alpha-f\mu_-)\,\{h_+\mu_+ + h_-\mu_- - f(h_+-h_-)\,\alpha\}} \tag{A 3b}$$

and

$$Z(\alpha) = \{G(\alpha)\,G(-\alpha)\}^{\frac{1}{2}}\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\log\left\{\frac{G(-t)}{G(t)}\right\}\frac{dt}{t-\alpha}\right]. \tag{A 4}$$

The result is independent of the choice of the square root in (A 4) provided that the same branch is used consistently. The integrals appear to be intractable without further approximation, such as an expansion about $\epsilon = 0$ (with first and second approximations equivalent to those given in § 4).

The solution for $|\epsilon f| > 1$ is complicated by the zeros of $G(\alpha)\,G(-\alpha)$ at $\alpha = \mp m$ (which tend to the real axis from above/below as $\delta \downarrow 0$), and eigensolutions must be added to the right-hand side of (A 2) in order to guarantee the appropriate behaviour of $\Phi(\alpha)$ as $\alpha \to \infty$. The end result is that the asymptotic ($x \to \infty$) solution contains a double Kelvin wave if and only if $\epsilon f > 1$, but the explicit calculation of the amplitude of this wave again leads to intractable integrals.

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